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Regularity of attractors for the Benjamin–Bona–Mahony equation

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Abstract. This paper deals with the regularity of the global attractor for the Benjamin–Bona–Mahony equation. We prove that the global attractor is smooth if the forcing term is smooth.

1. Introduction

In this paper, we investigate the asymptotic behaviour of solutions to the Benjamin–Bona–Mahony equation given by

$$u_t - u_{xxt} - \nu u_{xx} + u_x + uu_x = g(x) \qquad \text{in } \Omega \times R^+ \tag{1.1}$$

where $\nu > 0$, $g(x) \in L^2(\Omega)$, $\Omega \subset R$ is a bounded interval.

The Benjamin–Bona–Mahony equation incorporates nonlinear dispersive and dissipative effects, and has been proposed as a model for propagation of long waves. The existence and uniqueness of solutions for this equation have been investigated by many authors, such as Bona and Dougalis [1], Bona and Smith [2], Showalter [3] and Amick *et al* [4]. In the case $\nu = 0$, this equation has been studied by Benjamin *et al* [5], Bona and Bryant [6], Medeiros and Miranda [7], Medeiros and Menzala [8], Albert [9], Biler [10], and the references therein. The finite-dimensional behaviour of the solutions has been discussed by Wang and Yang [11], and Wang [12, 13]. In [11] and [12], the authors establish the existence of the global attractor for this model in $H^1(\Omega)$ and $H^2(\Omega)$, respectively, which has finite fractal dimension. In [13], the author proves the existence of the weak global attractor $\mathcal{A}_j$ in $H^j(\Omega)$ for every integer $j \geqslant 2$. Then a natural question arises whether these attractors coincide. This question is concerned with the regularity of attractors. The study of regularity of the global attractor is classical in the theory of dissipative dynamical systems, see [14] for the general framework which applies to the strongly dissipative case. For the weakly dissipative case such as the Benjamin–Bona–Mahony equation, the regularity of attractors is not obvious since, in this case, the equation has no regularization effect on the solutions. Recently, Goubet [15] apply a splitting method and successfully show the regularity of attractors for the weakly damped Schrödinger equation. Splitting techniques are also used to prove the regularity of attractors for weakly dissipative evolution equations by Moise and

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Rosa [16] and Goubet and Moise [17]. We here first construct a similar decomposition for the Benjamin–Bona–Mahony equation, and then show that $\mathcal{A}_j$ $(1 \leqslant j \leqslant k+2)$ obtained in [13] is a subset of $H^{k+2}(\Omega)$ if the forcing term $g \in H^k(\Omega)$ with $k \geqslant 0$. This result implies that $\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_{k+2}$.

The study of the asymptotic behaviour of solutions for nonlinear evolution equations is an interesting question both in mathematics and physics. For some dissipative equations, the long time behaviour of solutions is described by the existence of the global attractor which is a compact invariant set and attracts all solutions. The global attractor is closely related to the turbulence and chaos in physics. The complicated structure of the global attractor is one important cause of the perceived chaos (see [14]). So, to better understand the turbulence and chaos, it is necessary to investigate the existence and properties of attractors. In this paper, for the Benjamin–Bona–Mahony equation, the regularity of the global attractor is obtained. As a result of regularity, we will see that if the forcing term is smooth, then the global attractor is also smooth, which means the global attractor consists of smooth functions. In addition, this regularity also implies that the global attractor attracts all solutions in a stronger sense if the initial data are more regular. In practical problems, when we approach the orbits in the global attractor, the regularity will provide a higher order of approximation (see section 3 later).

The outline of this paper is as follows. In the next section, we present some known results for the Benjamin–Bona–Mahony equation which will be used in the following. Section 3 is devoted to our main result. We first introduce a splitting of the solution $u = S(t)u_0$ of equation (1.1) into two parts $S_1(t)u_0$ and $S_2(t)u_0$. Next, we derive *a priori* estimates and show that $S_1(t)u_0$ is more regular than $S(t)u_0$. Then, we prove that $S_2(t)u_0$ converges to zero when t goes to infinity. Finally, we present the proof of the main result.

2. Preliminaries

In this section, we describe the equation and recall some known results concerning the existence and uniqueness of solutions. We also state the existence of weak attractors and bounded absorbing sets for the equation.

We consider the following Benjamin–Bona–Mahony equation:

$$u_t - u_{xxt} - \nu u_{xx} + u_x + uu_x = g(x) \qquad (x,t) \in \Omega \times R^+ \tag{2.1}$$

with the initial condition

$$u(x,0) = u_0(x) \qquad x \in \Omega \tag{2.2}$$

and the periodic boundary condition

$$\Omega = (0, L) \qquad \text{and} \qquad u \text{ is } \Omega\text{-periodic} \tag{2.3}$$

where ν is a positive constant and $g(x)$ is a given function.

In the following, we shall denote $H = L^2_{\mathrm{per}}(\Omega)$, endowed with its usual inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, while $\|\cdot\|_p$ denotes the norm of $L^p_{\mathrm{per}}(\Omega)$ for all $1 \leqslant p \leqslant \infty (\|\cdot\|_2 = \|\cdot\|)$. In general, $\|\cdot\|_X$ denotes the norm of any Banach space X.

Throughout this paper, we assume $g \in H^k_{\mathrm{per}}(\Omega)$ with $k \geqslant 0$. Then it follows from [13] that, for every integer j with $1 \leqslant j \leqslant k+2$, if $u_0 \in H^j_{\mathrm{per}}(\Omega)$, the problem (2.1)–(2.3) possesses a unique solution $u(t)$ defined on R^+ such that

$$u(t) \in C([0,\infty), H^j_{\mathrm{per}}(\Omega)) \qquad \frac{\partial u}{\partial t} \in L^\infty(0,T; H^j_{\mathrm{per}}(\Omega)) \qquad \forall T > 0.$$

This result establishes the existence of a dynamical system $\{S^{(j)}(t)\}_{t\geqslant 0}$ which maps $H^j_{\mathrm{per}}(\Omega)$ to $H^j_{\mathrm{per}}(\Omega)$ such that $S^{(j)}(t)u_0 = u(t)$, the solution of problem (2.1)–(2.3). Clearly, $S^{(j)}(t)|_{H^{j+1}_{\mathrm{per}}} = S^{(j+1)}(t)$ for each $j = 1, 2, \ldots, k+1$. The weak continuity of $S^{(j)}(t)$ with respect to initial data is established in [11] and [12] for $1 \leqslant j \leqslant k+2$. That is, we have the following proposition.

Proposition 2.1. Assume that $g \in H^k_{\mathrm{per}}(\Omega)$ with $k \geqslant 0$. Then for each $j = 1, 2, \ldots, k+2$, and $t \geqslant 0$, $S^{(j)}(t) : H^j_{\mathrm{per}}(\Omega) \to H^j_{\mathrm{per}}(\Omega)$ is weakly continuous.

In this paper, we also suppose the forcing g has zero mean, that is,

$$\int_\Omega g(x)\,\mathrm{d}x = 0. \tag{2.4}$$

Then integrating (2.1) over Ω and applying (2.3) we find that the average of $u(t)$ is conserved, i.e. for all $t > 0$

$$\theta(u(t)) = \frac{1}{|\Omega|}\int_\Omega u(x,t)\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega u_0(x)\,\mathrm{d}x = \theta(u_0). \tag{2.5}$$

This shows that problem (2.1)–(2.3) has not bounded absorbing sets in whole space H. This difficulty is overcome by introducing

$$H_\alpha = \{u \in H : |\theta(u)| \leqslant \alpha\}.$$

(2.5) implies that H_α is invariant under the semigroup $S^{(j)}(t)$ associated to the system (2.1)–(2.3).

We now recall the following existence result of bounded absorbing sets.

Theorem 2.1. Assume that (2.4) holds, $g \in H^k_{\mathrm{per}}(\Omega)$ for a fixed $k \geqslant 0$, $u_0 \in H^j_{\mathrm{per}}(\Omega) \bigcap H_\alpha$, $j = 1, 2, \ldots, k+2$. Then there exists a constant E_j depending on the data (ν, Ω, g) and j such that

$$\|u(t)\|_{H^j} \leqslant E_j \qquad \forall t \geqslant t_j$$

where t_j depends on the data (ν, Ω, g) and j and R when $\|u_0\|_{H^j} \leqslant R$.

For the proof of this theorem, we refer the reader to [13].

We note that theorem 2.1 shows that the ball

$$B_j = \{u \in H^j_{\mathrm{per}}(\Omega) : \|u\|_{H^j} \leqslant E_j\} \tag{2.6}$$

is an absorbing set for $S^{(j)}(t)$ in $H^j_{\mathrm{per}}(\Omega) \bigcap H_\alpha$. Let

$$\mathcal{A}_j = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s} S(t)B_j} \qquad 1 \leqslant j \leqslant k+2 \tag{2.7}$$

where the closure is taken with respect to the H^j-weak topology. Then from [13] we know that $\mathcal{A}_j$ is the weak global attractor for $S^{(j)}(t)$. More precisely, we have the following.

Theorem 2.2. Assume that (2.4) holds, $g \in H^k_{\mathrm{per}}(\Omega)$ for a fixed $k \geqslant 0$, then for $j = 1, 2, \ldots, k+2$, the set $\mathcal{A}_j$ satisfies:

(i) $\mathcal{A}_j$ is bounded and weakly closed in $H^j_{\mathrm{per}}(\Omega) \cap H_\alpha$;

(ii) $S^{(j)}(t)\mathcal{A}_j = \mathcal{A}_j$, $\forall t \geqslant 0$;

(iii) for every bounded set B in $H^j_{\mathrm{per}} \cap H_\alpha$, $S^{(j)}(t)B$ converges to $\mathcal{A}_j$ with respect to the H^j-weak topology as $t \to \infty$.

We remark that the weak global attractors $\mathcal{A}_1$ and $\mathcal{A}_2$ are also the strong global attractors in $H^1_{\mathrm{per}}(\Omega)$ and $H^2_{\mathrm{per}}(\Omega)$, respectively, which is shown in [11] and [12]. Here the strong global attractor means that the conclusion (iii) in theorem 2.2 holds with respect to the strong topology.

By theorem 2.2, it is easy to see that $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots \supset \mathcal{A}_{k+2}$. In this paper, we shall show also $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_{k+2}$. So we will find that $\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_{k+2}$. For that purpose, we must establish the regularity of $\mathcal{A}_j$ for $j = 1, 2, \dots, k+1$, see theorem 3.2 below.

In the following, we will frequently use the Agmon inequality: for $u \in H^1(\Omega)$,

$$\|u\|_\infty \leqslant C\|u\|^{1/2}\|u\|^{1/2}_{H^1} \leqslant C\|u\|_{H^1}. \tag{2.8}$$

Hereafter, we shall denote by C any positive constants which may change from line to line.

3. Regularity of the global attractor

In this section, we shall use a decomposition technique to derive the regularity of the weak global attractor $\mathcal{A}_j$ for $S^{(j)}(t)$, and show that all the attractors $\mathcal{A}_j$ coincide. By an idea of Ball [18], we shall also prove that the weak global attractor $\mathcal{A}_j$ is actually the strong global attractor for $S^{(j)}(t)$ in $H^j_{\mathrm{per}}(\Omega)$.

In what follows, we denote by $A = -\partial_{xx}$, the unbounded self-adjoint operator with domain $H^2_{\mathrm{per}}(\Omega)$. Then the operator $A^{1/2}$ is well defined. By spectral theory, we know that there exists a complete orthonormal basis $\{w_n\}_{n=1}^\infty$ of H consisting of eigenvectors of A, that is,

$$Aw_n = \lambda_n w_n \qquad 0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n \to \infty.$$

Given N, we denote by $P = P_N$ the orthogonal projectors in H onto the space spanned by the first N eigenvectors of A, $w_1, w_2, \dots, w_N$, and we set $Q = Q_N = I - P_N$. Since $\|A^{1/2}u\| = \|u_x\|$, for $u \in H^1_{\mathrm{per}}(\Omega)$, and

$$\|A^{1/2}u\| \geqslant \lambda^{1/2}_{N+1}\|u\| \qquad u \in Q_N D(A^{1/2})$$

we see that

$$\|u\| \leqslant \lambda^{-1/2}_{N+1}\|u_x\| \qquad u \in Q_N D(A^{1/2}). \tag{3.1}$$

Consider now $u_0 \in H^1_{\mathrm{per}}(\Omega)$, $u(t) = S^{(1)}(t)u_0$, and set $u(t) = p(t) + q(t)$, where $p(t) = P_N u(t)$, $q(t) = Q_N u(t)$. We split the high-frequency part q as $q = y + z$, where y is defined by

$$y_t - y_{xxt} - \nu y_{xx} + y_x + Q_N(yy_x) + Q_N(py)_x = Q_N g(x) - Q_N(pp_x) \tag{3.2}$$

$$y(0) = 0. \tag{3.3}$$

The following lemma shows that problem (3.2)–(3.3) is well-posed.

Lemma 3.1. Assume $g \in H$ and $u_0 \in H^1_{\mathrm{per}}(\Omega)$. Then there exists a unique solution y of problem (3.2)–(3.3) such that $y \in C([0,\infty), Q_N H^1_{\mathrm{per}}(\Omega))$.

The proof of this lemma is similar to that in [7], and therefore is omitted here.

In the following, we derive estimates on y and show that $y(t)$ is more regular than $u(t)$. For that purpose, we will use the bounded absorbing set B_j $(1 \leqslant j \leqslant k+2)$ given by (2.6). Without loss of generality, we can assume that $S^{(j)}(t)B_j \subset B_j$ for every $t \geqslant 0$, which means if $u_0 \in B_j$ $(1 \leqslant j \leqslant k+2)$, then for every $t \geqslant 0$,

$$\|u(t)\|_{H^j} \leqslant E_j. \tag{3.4}$$

By (3.4) and the Agmon inequality we see that, for every $t \geqslant 0$,

$$\|u(t)\|_\infty \leqslant C. \tag{3.5}$$

Due to $p(t) = P_N u(t)$, by (3.4) we have for every $t \geqslant 0$,

$$\|p(t)\|_{H^j} \leqslant E_j \qquad \|p(t)\|_\infty \leqslant C. \tag{3.6}$$

We now derive the estimates on y in $H^1_{\mathrm{per}}(\Omega)$.

Lemma 3.2. Assume that (2.4) holds, $g \in H$, $u_0 \in B_1$. Then there exists N_0 depending on E_1 such that for $N \geqslant N_0$,

$$\|y(t)\|_{H^1} \leqslant C \qquad t \geqslant 0$$

where C depends on the data (ν, Ω, g).

Proof. Taking the inner product of (3.2) with y in H, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|y\|^2 + \|y_x\|^2) + \nu\|y_x\|^2 = (Qg, y) - (Q(pp_x + yy_x + (py)_x), y). \tag{3.7}$$

By (3.1) we have

$$|(Qg, y)| = |(g, y)| \leqslant \|g\|\|y\| \leqslant \lambda_{N+1}^{-1/2}\|g\|\|y_x\| \leqslant C + \lambda_{N+1}^{-1}\|y_x\|^2. \tag{3.8}$$

By (3.6) with $j = 1$, we get

$$\begin{aligned} |(-Q(pp_x), y)| = |(pp_x, y)| &= \left|\int_\Omega pp_x g \,\mathrm{d}x\right| \leqslant \|p\|_\infty\|p_x\|\|y\| \leqslant C\|y\| \\ &\leqslant C + \|y\|^2 \leqslant C + \lambda_{N+1}^{-1}\|y_x\| \end{aligned} \tag{3.9}$$

while

$$-(Q(yy_x), y) = -(yy_x, y) = -\frac{1}{3}\int_\Omega (y^3)_x \,\mathrm{d}x = 0. \tag{3.10}$$

By (3.6) again, we have

$$-(Q(py)_x, y) = (py, y_x) \leqslant \|p\|_\infty\|y\|\|y_x\| \leqslant C\lambda_{N+1}^{-1/2}\|y_x\|^2. \tag{3.11}$$

Without loss of generality, we always assume $\lambda_{N+1} \geqslant 1$ in the following. So by (3.7)–(3.11) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|y\|^2 + \|y_x\|^2) + 2\nu\|y_x\|^2 \leqslant C + C_1\lambda_{N+1}^{-1/2}\|y_x\|^2. \tag{3.12}$$

Choosing N_0 large enough such that $C_1\lambda_{N_0+1}^{-1/2} \leqslant \nu$. Then for $N \geqslant N_0$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|y\|^2 + \|y_x\|^2) + \nu\|y_x\|^2 \leqslant C. \tag{3.13}$$

By (3.1), we note that

$$\|y_x\|^2 \geqslant \tfrac{1}{2}\|y_x\|^2 + \tfrac{1}{2}\lambda_{N+1}\|y\|^2 \geqslant \tfrac{1}{2}(\|y\|^2 + \|y_x\|^2)$$

and therefore, we find, for every $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|y\|^2 + \|y_x\|^2) + \frac{1}{2}\nu(\|y\|^2 + \|y_x\|^2) \leqslant C.$$

It follows from the Gronwall lemma that, for every $t \geqslant 0$,

$$\|y(t)\|^2 + \|y_x(t)\|^2 \leqslant \mathrm{e}^{-\frac{1}{2}\nu t}(\|y(0)\|^2 + \|y_x(0)\|^2) + \frac{2C}{\nu} = \frac{2C}{\nu}$$

which concludes lemma 3.2. $\square$

By lemma 3.2 and the Agmon inequality, we get, for all $t \geqslant 0$,

$$\|y(t)\|_\infty \leqslant C. \tag{3.14}$$

Lemma 3.3. Assume that (2.4) holds, $g \in H$, $u_0 \in B_1$. Then, for $N \geqslant N_0$, we have

$$\|y(t)\|_{H^2} \leqslant C \qquad t \geqslant 0$$

where N_0 is the constant in lemma 3.2, C depends on the data (ν, Ω, g).

Proof. Taking the inner product of (3.2) with $-y_{xx}$ in H, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|y_x\|^2 + \|y_{xx}\|^2) + \nu\|y_{xx}\|^2 = (g - pp_x - yy_x - (py)_x, -y_{xx}). \tag{3.15}$$

Note that

$$|(g, -y_{xx})| \leqslant \|g\|\|y_{xx}\| \leqslant C + \tfrac{1}{8}\nu\|y_{xx}\|^2. \tag{3.16}$$

By (3.6) we have

$$|(pp_x, y_{xx})| \leqslant \|p\|_\infty\|p_x\|\|y_{xx}\| \leqslant C\|y_{xx}\| \leqslant C + \tfrac{1}{8}\nu\|y_{xx}\|^2. \tag{3.17}$$

By (3.14) and lemma 3.2, we get

$$|(yy_x, y_{xx})| \leqslant \|y\|_\infty\|y_x\|\|y_{xx}\| \leqslant C\|y_{xx}\| \leqslant C + \tfrac{1}{8}\nu\|y_{xx}\|^2. \tag{3.18}$$

Similarly, we also have

$$|((py)_x, y_{xx})| \leqslant C + \tfrac{1}{8}\nu\|y_{xx}\|^2. \tag{3.19}$$

By (3.15)–(3.19), we get, for all $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|y_x\|^2 + \|y_{xx}\|^2) + \nu\|y_{xx}\|^2 \leqslant C.$$

Since

$$\|y_{xx}\|^2 \geqslant \tfrac{1}{2}\|y_{xx}\|^2 + \tfrac{1}{2}\lambda_{N+1}\|y_x\|^2 \geqslant \tfrac{1}{2}(\|y_{xx}\|^2 + \|y_x\|^2)$$

we find, for all $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|y_x\|^2 + \|y_{xx}\|^2) + \tfrac{1}{2}\nu(\|y_x\|^2 + \|y_{xx}\|^2) \leqslant C$$

and then the Gronwall lemma gives lemma 3.3. □

Lemma 3.4. Assume that (2.4) holds, $g \in H^k_{\mathrm{per}}(\Omega)$ for a fixed $k \geqslant 0$, $u_0 \in B_j$ $(1 \leqslant j \leqslant k+1)$. Then there exists a constant C_j depending on the data (ν, Ω, g) and j such that when $N \geqslant N_0$,

$$\|y(t)\|_{H^{j+1}} \leqslant C_j \qquad t \geqslant 0$$

where N_0 is the constant in lemma 3.2.

Proof. We check this lemma by an induction argument on j.

(i) Initialization of the induction $(j = 1)$. If $j = 1$, lemma 3.4 reduces to lemma 3.3. Therefore, in this case, the lemma is true.

(ii) The induction argument. We assume that lemma 3.4 holds up to order $j - 1$; we want to show it is also valid at order j $(j \geqslant 2)$.

Taking the inner product of (3.2) with $(-1)^j\partial_x^{2j}y$ in H, we see

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_x^j y\|^2 + \|\partial_x^{j+1}y\|^2) + \nu\|\partial_x^{j+1}y\|^2 = (-1)^j(g - pp_x - yy_x - (py)_x, \partial_x^{2j}y). \tag{3.20}$$

By the induction assumption, we know

$$\|y(t)\|_{H^j} \leqslant C_j \qquad t \geqslant 0. \tag{3.21}$$

So by the Agmon inequality, we have, for each $0 \leqslant l \leqslant j-1$,

$$\|\partial_x^l y(t)\|_\infty \leqslant C_j \qquad t \geqslant 0. \tag{3.22}$$

Note that

$$|(-1)^j(g, \partial_x^{2j} y)| \leqslant \|g\|_{H^{j-1}}\|\partial_x^{j+1} y\| \leqslant \|g\|_{H^k}\|\partial_x^{j+1} y\| \leqslant C + \tfrac{1}{8}\nu\|\partial_x^{j+1}\|^2. \tag{3.23}$$

We also have

$$\begin{aligned}
|(-1)^j(-pp_x, \partial_x^{2j} y)| &= \left|\int_\Omega \partial_x^{j-1}(pp_x)\partial_x^{j+1} y \,\mathrm{d}x\right| \\
&\leqslant \left|\int_\Omega \sum \alpha_j (\partial_x p)^{a_1} \ldots (\partial_x^{j-1} p)^{a_{j-1}} \partial_x^{k+2} y \,\mathrm{d}x\right| \\
&\quad + \left|\int_\Omega p \partial_x^j p \partial_x^{j+1} y \,\mathrm{d}x\right| \leqslant \sum \alpha_j \|\partial_x p\|_\infty^{a_1} \ldots \|\partial_x^{j-1} p\|_\infty^{a_{j-1}} \int_\Omega |\partial_x^{j+1} y| \,\mathrm{d}x \\
&\quad + \|p\|_\infty \|\partial_x^j p\| \|\partial_x^{j+1} y\| \leqslant \sum \alpha_j |\Omega|^{1/2} \|p\|_{H^2}^{a_1} \ldots \|p\|_{H^j}^{a_{j-1}} \|\partial_x^{j+1} y\| \\
&\quad + \|p\|_{H^1} \|p\|_{H^j} \|\partial_x^{j+1} y\| \leqslant C\|\partial_x^{j+1} y\| \text{ (by (3.6))} \\
&\leqslant C + \frac{1}{8}\nu\|\partial_x^{j+1} y\|^2.
\end{aligned} \tag{3.24}$$

Similarly, using (3.21) and (3.22) instead of (3.6), we can deduce that the last two terms on the right-hand side of (3.20) are also bounded by $C + \frac{1}{8}\nu\|\partial_x^{j+1} y\|^2$, so by (3.20), (3.23), (3.24) and the analogy, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_x^j y\|^2 + \|\partial_x^{j+1} y\|^2) + \nu\|\partial_x^{j+1} y\|^2 \leqslant C.$$

Thanks to

$$\|\partial_x^{j+1} y\|^2 \geqslant \tfrac{1}{2}\|\partial_x^{j+1} y\|^2 + \tfrac{1}{2}\lambda_{N+1}\|\partial_x^j y\|^2 \geqslant \tfrac{1}{2}(\|\partial_x^{j+1} y\|^2 + \|\partial_x^j y\|^2)$$

we get, for all $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_x^j y\|^2 + \|\partial_x^{j+1} y\|^2) + \frac{1}{2}\nu(\|\partial_x^j y\|^2 + \|\partial_x^{j+1} y\|^2) \leqslant C$$

and then the Gronwall lemma concludes lemma 3.4. □

Projecting equation (2.1) onto $Q_N H$, we find that $q = Q_N u$ satisfies

$$q_t - q_{xxt} - \nu q_{xx} + q_x + Q_N(p+q)(p+q)_x = Q_N g \tag{3.25}$$

$$q(0) = Q_N u_0. \tag{3.26}$$

Then from (3.25) and (3.2), it follows that $z = q - y$ satisfies

$$z_t - z_{xxt} - \nu z_{xx} + z_x = -Q_N(zq_x + yz_x + (pz)_x) \tag{3.27}$$

$$z(0) = Q_N u_0. \tag{3.28}$$

In the following, we shall show z converges to zero as t goes to infinity. More precisely, we have the following.

Lemma 3.5. Assume that (2.4) holds, $g \in H$, $u_0 \in B_1$. Then there exists N_0 depending on the data (ν, Ω, g) and E_1 such that, for $N \geqslant N_0$,

$$\|z(t)\|_{H^1}^2 \leqslant E_1 \,\mathrm{e}^{-\frac{1}{2}\nu t} \qquad t \geqslant 0$$

where E_1 is the constant in (2.6) with $j = 1$.

Proof. Taking the inner product of (3.27) with z in H, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|z\|^2 + \|z_x\|^2) + \nu\|z_x\|^2 = -(zq_x + yz_x + (pz)_x, z). \tag{3.29}$$

We now majorize each term in (3.29) as follows. By (3.1) and the Agmon inequality, we get

$$\begin{aligned}(-zq_x, z) &= 2\int_\Omega qzz_x \,\mathrm{d}x \leqslant 2\|q\|_\infty\|z\|\|z_x\| \leqslant C\lambda_{N+1}^{-1/2}\|q\|_{H^1}\|z_x\|^2 \leqslant C\lambda_{N+1}^{-1/2}\|u\|_{H^1}\|z_x\|^2 \\ &\leqslant C\lambda_{N+1}^{-1/2}\|z_x\|^2.\end{aligned} \tag{3.30}$$

Due to

$$-(yz_x + (pz)_x, z) = -\int_\Omega yzz_x \,\mathrm{d}x + \int_\Omega pzz_x \,\mathrm{d}x$$

we can see that the above is also bounded by $C\lambda_{N+1}^{-1/2}\|z_x\|^2$. Then we get, from (3.29), (3.30) and the analogy, that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|z\|^2 + \|z_x\|^2) + 2\nu\|z_x\|^2 \leqslant C\lambda_{N+1}^{-1/2}\|z_x\|^2 \qquad t \geqslant 0.$$

Choosing N_0 large enough such that $C\lambda_{N_0+1}^{-1/2} \leqslant \nu$, we find that, for $N \geqslant N_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|z\|^2 + \|z_x\|^2) + \nu\|z_x\|^2 \leqslant 0.$$

Again, by

$$\|z_x\|^2 \geqslant \tfrac{1}{2}\|z_x\|^2 + \tfrac{1}{2}\lambda_{N+1}\|z\|^2 \geqslant \tfrac{1}{2}(\|z\|^2 + \|z_x\|^2)$$

we see, for all $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|z\|^2 + \|z_x\|^2) + \tfrac{1}{2}\nu(\|z\|^2 + \|z_x\|^2) \leqslant 0.$$

By the Gronwall lemma and $u_0 \in B_1$, we have, for all $t \geqslant 0$,

$$\begin{aligned}\|z(t)\|^2 + \|z_x(t)\|^2 &\leqslant \mathrm{e}^{-\frac{1}{2}\nu t}(\|z(0)\|^2 + \|z_x(0)\|^2) \\ &\leqslant \mathrm{e}^{-\frac{1}{2}\nu t}(\|Q_N u(0)\|^2 + \|Q_N u_x(0)\|^2) \leqslant E_1\,\mathrm{e}^{-\frac{1}{2}\nu t}\end{aligned}$$

which concludes lemma 3.5. □

In the following, we shall show that, for every $j = 1, 2, \ldots, k+2$, the weak global attractor $\mathcal{A}_j$ is actually the strong global attractor for $S^{(j)}(t)$ in $H^j_{\mathrm{per}}(\Omega)$, which will be proved by an idea due to Ball [18].

Theorem 3.1. Assume that (2.4) holds, $g \in H^k_{\mathrm{per}}(\Omega)$ for a fixed $k \geqslant 0$. Then for every $j = 1, 2, \ldots, k+2$, the weak global attractor $\mathcal{A}_j$ is actually the strong global attractor in $H^j_{\mathrm{per}}(\Omega)$.

Proof. The proof of this theorem is similar to that in [11], so here we only sketch it.

Since a point w belongs to $\mathcal{A}_j$ if and only if there exist two sequences $\{w_m^0\}_{m=1}^\infty \subset B_j$ and $\{t_m\}_{m=1}^\infty$, $t_m \to \infty$, such that $S^{(j)}(t_m)w_m^0$ converges to w weakly in $H^j_{\mathrm{per}}(\Omega)$, this theorem will be proved if we are able to show that (some subsequence of) the sequence $S^{(j)}(t_m)w_m^0$ converges to w strongly in $H^j_{\mathrm{per}}(\Omega)$.

Taking the inner product of (2.1) with $(-1)^{j-1}\partial_x^{2(j-1)}u$ in H, we find that any solution u of problem (2.1)–(2.3) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_x^{j-1}u\|^2 + \|\partial_x^j u\|^2) + 2\nu(\|\partial_x^{j-1}u\|^2 + \|\partial_x^j u\|^2) = K(u)$$

where

$$K(u) = 2\nu\|\partial_x^{j-1}u\|^2 + 2\int_\Omega \partial_x^{j-2}(uu_x)\cdot\partial_x^j u\,\mathrm{d}x - 2\int_\Omega \partial_x^{j-2}g\cdot\partial_x^j u\,\mathrm{d}x.$$

We observe that $K(u)$ is weakly continuous in $H^j_{\mathrm{per}}(\Omega)$. Then by proposition 2.1, using the technique of [18] and proceeding as in [11], we can deduce that

$$\limsup_{m\to\infty}(\|\partial_x^{j-1}(S^{(j)}(t_m)w_m^0)\|^2 + \|\partial_x^j(S^{(j)}(t_m)w_m^0)\|^2 \leqslant \|\partial_x^{j-1}w\|^2 + \|\partial_x^j w\|^2. \tag{3.31}$$

Since $S^{(j)}(t_m)w_m^0 \to w$ weakly in $H^j_{\mathrm{per}}(\Omega)$, by the Sobolev imbedding theorem, we see $S^{(j)}(t_m)w_m^0 \to w$ strongly in $H^{j-1}_{\mathrm{per}}(\Omega)$, up to a subsequence. Therefore, by (3.31), we get

$$\limsup_{m\to\infty}\|S^{(j)}(t_m)w_m^0\|^2_{H^j} \leqslant \|w\|^2_{H^j}. \tag{3.32}$$

On the other hand, by weak convergence, we have

$$\liminf_{m\to\infty}\|S^{(j)}(t_m)w_m^0\|^2_{H^j} \geqslant \|w\|^2_{H^j}. \tag{3.33}$$

(3.32) and (3.33) along with the weak convergence imply that $S^{(j)}(t_m)w_m^0 \to w$ strongly in $H^j_{\mathrm{per}}(\Omega)$. Hence, we get that $\mathcal{A}_j$ is the strong global attractor in $H^j_{\mathrm{per}}(\Omega)$. The proof is complete. □

We now show that all the global attractors $\mathcal{A}_j$ coincide. For that purpose, we decompose the semigroup $S^{(j)}(t)$ defined on $H^j_{\mathrm{per}}(\Omega)$ as $S^{(j)}(t) = S_1^{(j)}(t) + S_2^{(j)}(t)$ for every $j = 1, 2, \ldots, k+2$. For $u_0 \in H^j_{\mathrm{per}}(\Omega)$, we define for all $t \geqslant 0$,

$$S_1^{(j)}(t)u_0 = p(t) + y(t) \qquad S_2^{(j)}(t)u_0 = z(t) \tag{3.34}$$

where $p(t) = P_N u(t) = P_N S^{(j)}(t)u_0$, $y(t)$ is the solution of problem (3.2)–(3.3) and $z(t)$ is the solution of problem (3.27) and (3.28). Clearly, we have, for every $j = 1, 2, \ldots, k+2$,

$$S^{(j)}(t) = S_1^{(j)}(t) + S_2^{(j)}(t) \qquad t \geqslant 0. \tag{3.35}$$

Our main result is as follows.

Theorem 3.2. Assume that (2.4) holds, $g \in H^k_{\mathrm{per}}(\Omega)$ for a fixed $k \geqslant 0$. Then we have

$$\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_{k+2}.$$

Proof. We only need to check, for every $j = 1, 2, \ldots, k+1$, $\mathcal{A}_j = \mathcal{A}_{j+1}$.

Given $w \in \mathcal{A}_j$, we know that there exist $w_n \in B_j$ and $t_n \to \infty$ such that

$$S^{(j)}(t_n)w_n \to w \qquad \text{in } H^j_{\mathrm{per}}(\Omega). \tag{3.36}$$

By (3.35), we also have

$$S^{(j)}(t_n)w_n = S_1^{(j)}(t_n)w_n + S_2^{(j)}(t_n)w_n. \tag{3.37}$$

By (3.34) and (3.6) and lemma 3.4, we see

$$\|S_1^{(j)}(t_n)w_n\|_{H^{j+1}} \leqslant C(1 + \lambda_N^{1/2}). \tag{3.38}$$

So there exist subsequences of w_n and t_n (still denoted by w_n and t_n) and $v \in H^{j+1}_{\mathrm{per}}(\Omega)$ such that

$$S_1^{(j)}(t_n)w_n \to v \qquad \text{weakly in } H^{j+1}_{\mathrm{per}}(\Omega). \tag{3.39}$$

Also, we have

$$\|v\|_{H^{j+1}} \leqslant \liminf_{n\to\infty} \|S_1^{(j)}(t_n)w_n\|_{H^{j+1}} \leqslant C(1+\lambda_N^{1/2}). \tag{3.40}$$

Consider now $\phi \in H$. From (3.37), we have

$$(S^{(j)}(t_n)w_n, \phi) = (S_1^{(j)}(t_n)w_n, \phi) + (S_2^{(j)}(t_n)w_n, \phi).$$

Then, taking the limit as $n \to \infty$, by (3.36) and (3.39) and lemma 3.5, we get $(w, \phi) = (v, \phi)$ for every $\phi \in H$, which implies $w = v$ in H. So we see $w \in H^{j+1}_{\mathrm{per}}(\Omega)$. From (3.40), we also have, for all $w \in \mathcal{A}_j$,

$$\|w\|_{H^{j+1}} \leqslant C(1+\lambda_N^{1/2}).$$

This means $\mathcal{A}_j$ is a bounded set in $H^{j+1}_{\mathrm{per}}(\Omega)$. Since $\mathcal{A}_{j+1}$ attracts every bounded set in $H^{j+1}_{\mathrm{per}}(\Omega)$, we have

$$\mathrm{dist}_{H^{j+1}}(\mathcal{A}_j, \mathcal{A}_{j+1}) = \mathrm{dist}_{H^{j+1}}(S^{(j+1)}(t)\mathcal{A}_j, \mathcal{A}_{j+1}) \to 0 \qquad \text{as } t \to \infty$$

which implies $\mathcal{A}_j \subset \mathcal{A}_{j+1}$. Obviously, $\mathcal{A}_{j+1} \subset \mathcal{A}_j$. Therefore, $\mathcal{A}_j = \mathcal{A}_{j+1}$ for all $j = 1, 2, \ldots, k+1$. The proof is complete. □

In what follows, we show that regularity of the global attractor is useful when we approach the solutions. In practical problems, we often need a finite-dimensional approximation to the orbits in the global attractor. A simple way to do this is to approximate the orbit $u(t)$ in $\mathcal{A}_1$ by its N-dimensional component $p(t)$ in the space spanned by the first N eigenvectors of A. In the following, we assume $g \in H$. Since $\mathcal{A}_1 \subset B_1$, it follows from (2.6) and (3.1) that the error $q(t) = u(t) - p(t)$ verifies

$$\|q(t)\| \leqslant \lambda_{N+1}^{-1/2}\|q(t)\|_{H^1} \leqslant \lambda_{N+1}^{-1/2}\|u(t)\|_{H^1} \leqslant E_1\lambda_{N+1}^{-1/2}. \tag{3.41}$$

On the other hand, by theorem 3.2 with $k = 0$ we have $u(t) \in \mathcal{A}_2 \subset B_2$. So, it follows that

$$\|q(t)\| \leqslant \lambda_{N+1}^{-1}\|q(t)\|_{H^2} \leqslant \lambda_{N+1}^{-1}\|u(t)\|_{H^2} \leqslant E_2\lambda_{N+1}^{-1}. \tag{3.42}$$

Noting $\lambda_{N+1} \to +\infty$ as $N \to \infty$, we see that (3.42) provides a better approximation than (3.41) when N is large enough. In other words, if we want to get the same error, then the dimension N provided by (3.42) is smaller than that by (3.41).

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